

## **General Disclaimer**

### **One or more of the Following Statements may affect this Document**

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.
- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.
- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.
- This document is paginated as submitted by the original source.
- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.



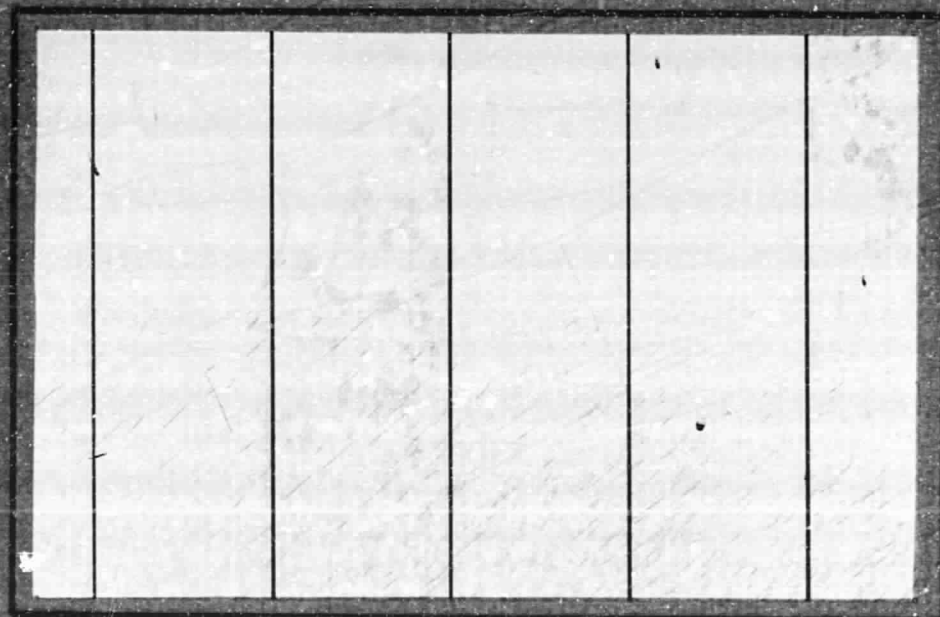
# OF COLLEGE ENGINEERING

N77-23539

Unclas  
29049

(NASA-CR-153047) A METHOD FOR EXPERIMENTAL  
MODAL SEPARATION Final Technical Report,  
Feb. 1976 - May 1977 (Virginia Polytechnic  
Inst. and State Univ.) 24 p HC A02/MF A01

CSCL 20K G3/39



**VIRGINIA  
POLYTECHNIC  
INSTITUTE  
AND  
STATE  
UNIVERSITY**

**BLACKSBURG,  
VIRGINIA**



**FINAL TECHNICAL REPORT**

**February 1976 - May 1977**

**NASA Langley Research Center  
Grant No. MSG 1276**

**A METHOD FOR EXPERIMENTAL MODAL SEPARATION**

**William L. Hallauer, Jr.  
Aerospace and Ocean Engineering Department  
Virginia Polytechnic Institute and State University  
Blacksburg, Virginia 24061**



FINAL TECHNICAL REPORT  
A METHOD FOR EXPERIMENTAL MODAL SEPARATION\*

I. Introduction

A system of computer software has been developed for the numerical simulation of multiple-shaker modal survey testing. A fairly simple analytical method using simulated experimental data is employed to optimize the shaker force-amplitude distribution for the purpose of isolating individual modes of vibration. This method is referred to here as the RFAD method, for relative force-amplitude distribution.

The system has been employed in a thorough simulation of modal testing on a small, fairly simple, analytical structural model. The detailed description of this simulation is included in a Master's degree thesis by Stafford (1976). The software system is currently being used to test the RFAD method on various other structural models, both discrete and continuous. Each model was designed to have at least two closely spaced modes in order that we might simulate the separation of close modes by multiple-shaker excitation. Inertia, damping, stiffness, and modal data for these models are stored on magnetic disks, available by direct access to the interactive Fortran programs which perform all computations required by the RFAD method.

The support for this research by NASA has been renewed for an additional year under the same number, Grant No. NSG 1276. Hence, this report is not the final presentation of theory, procedures and results to be submitted on our research into the RFAD method.

---

\* The NASA Technical Officer for this grant is Mr. Robert Miserentino, NASA Langley Research Center.



## II. Short Description of the RFAD Method

The RFAD method is based on the following theoretical result for an  $n$ -degree-of-freedom damped structure. In order that the structure be forced to vibrate in its  $r$ th undamped normal mode, the condition

$$[C(\omega_r)] \underline{F} = \underline{0}$$

must be satisfied (Asher, 1958; Bishop and Gladwell, 1963; Craig and Su, 1974).  $\underline{F}$  is the  $n \times 1$  column vector of shaker force-amplitudes, and  $[C(\omega_r)]$  is the  $n \times n$  coincident-response matrix\* evaluated at the  $r$ th resonance frequency,  $\omega_r$ . In practice, it is generally not possible to provide forcing excitation at all the degrees of freedom of a structure, so the problem which the RFAD method considers is

$$[C^*] \underline{F}^* = \underline{0}$$

where  $\underline{F}^*$  is a  $p \times 1$  matrix of force-amplitudes corresponding to the  $p$  available shaker locations, and  $[C^*]$  is the incomplete  $p \times p$  coincident-response matrix for those same locations. This is an eigenvalue problem for which, ideally, the eigenvalues (those frequencies at which  $\text{Det } [C^*] = 0$ ) are the undamped natural frequencies, and the eigenvectors are the force-amplitude distributions which will tune reasonably pure modes.

## III. Analytical Models with Closely Spaced Modes

Perhaps the most difficult task in modal survey testing is the separation of

---

\* The coincident-response matrix is the real or in-phase portion of the complete frequency-response transfer function matrix.



closely spaced modes. Hence, one of our primary objectives in developing analytical models to test with the RFAD method has been to simulate closely spaced modes. Although such modes appear frequently in vehicle modal testing programs, they are actually quite difficult to produce in relatively simple structural models of the type appropriate for this research. Hence, we have developed an analytical procedure for "optimizing" a structural model to have at least one pair of closely spaced modes.

The structural model we have considered most extensively is the five-degree-of-freedom cantilevered plane grid shown in Fig. 1. The beams are aluminum rods having 0.254 m (10.0 inch) length\* and 0.00635 m (0.25 inch) diameter;

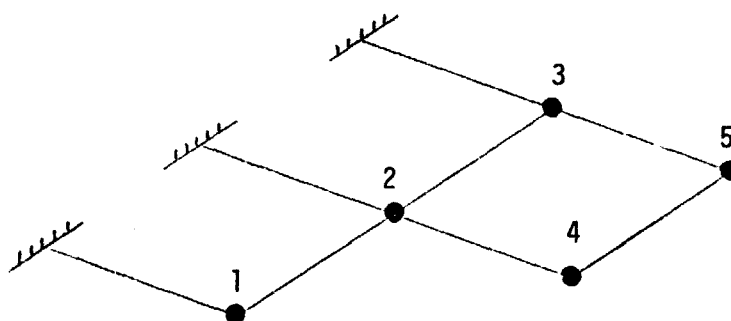


Figure 1 Plane Grid Model

they are considered massless. All mass is considered to be concentrated at the beam intersections. Our procedure has been to adjust the concentrated masses in order to force together selected adjacent modes.

---

\* All calculations have been made in the primary units of pounds, inches and seconds.



Stafford (1976) used an early form of our analytical mode-merging method to produce closely spaced third and fourth modes of this model having frequencies of 30.54 rad/sec and 31.90 rad/sec, respectively. However, this situation resulted from a set of masses weighing from 11.9 N (2.68 lb.) to 114 N (25.7 lb.). In view of the small size of the model, the latter value is unrealistically large. But we try to retain a reasonable degree of physical realism in our analytical models, since we may elect later to build laboratory specimens based on them. Hence, we have refined our mode-merging procedure so as to hold the masses to physically realistic values. The theory and procedure are presented in Appendix A. With this procedure we have produced an eigensolution for the five-degree-of-freedom model having third and fourth resonances of 53.50 rad/sec and 55.41 rad/sec, respectively. The masses in this model weigh from 3.38 N (0.760 lb.) to 20.6 N (4.62 lb.).

#### IV. Simulation of Arbitrary Hysteretic Damping

We have studied primarily n-degree-of-freedom damped models governed by the linear matrix equations

$$[m] \ddot{\underline{x}} + \frac{1}{\omega} [d] \dot{\underline{x}} + [k] \underline{x} = \underline{f}$$

where  $[d]$  is the hysteretic damping matrix, and excitation and response are sinusoidal at frequency  $\omega$ ,

$$\underline{f} = \underline{F}e^{i\omega t}$$

$$\underline{x} = \underline{X}e^{i\omega t}$$

All previous studies of the RFAD method (Bishop and Gladwell, 1963; Craig and Su, 1974; Stafford, 1976) have considered only the case of hysteretic



damping which does not couple the undamped normal modes of the system.

Stafford used a proportional damping matrix,

$$[d] = g [k]$$

which is uncoupled by the standard transformation to normal coordinates and gives structural damping constant  $g$  in all modes.

However, there is no reason to expect in general that damping will not couple the modes. So we have developed the capability to simulate a structure with arbitrary hysteretic damping. The theory behind this simulation is evidently due to Bishop and Johnson (1960), and it has been discussed extensively by Mead (1970). It involves first determination of the complex eigenvalues  $\lambda_r$  and eigenvectors  $\underline{\psi}_r$  of the homogeneous problem

$$([k] + i [d] - \lambda [m]) \underline{\psi} = 0$$

and then solution for the steady-state response due to sinusoidal excitation by modal analysis with the complex modes.

To provide an example of the possible effects of damping which couples the normal modes, we have applied the RFAD method with both proportional and non-proportional damping to the five-degree-of-freedom model discussed in Section III. For the proportional case we use  $[d] = 0.03 [k]$ , and for the non-proportional case we use simply the diagonal portion of the proportional damping matrix. Since a primary objective of the RFAD method is to separate closely spaced modes, we seek distributions of shaker force-amplitudes which will separate the third and fourth modes at 53.50 and 55.41 rad/sec, respectively

Plots of  $\text{Det } [C(\omega)]$  for all five masses excited and  $\text{Det } [C^*(\omega)]$  for only masses 1 and 5 (see Fig. 1) excited are shown on Fig. 2.  $\text{Det } [C]$  curves



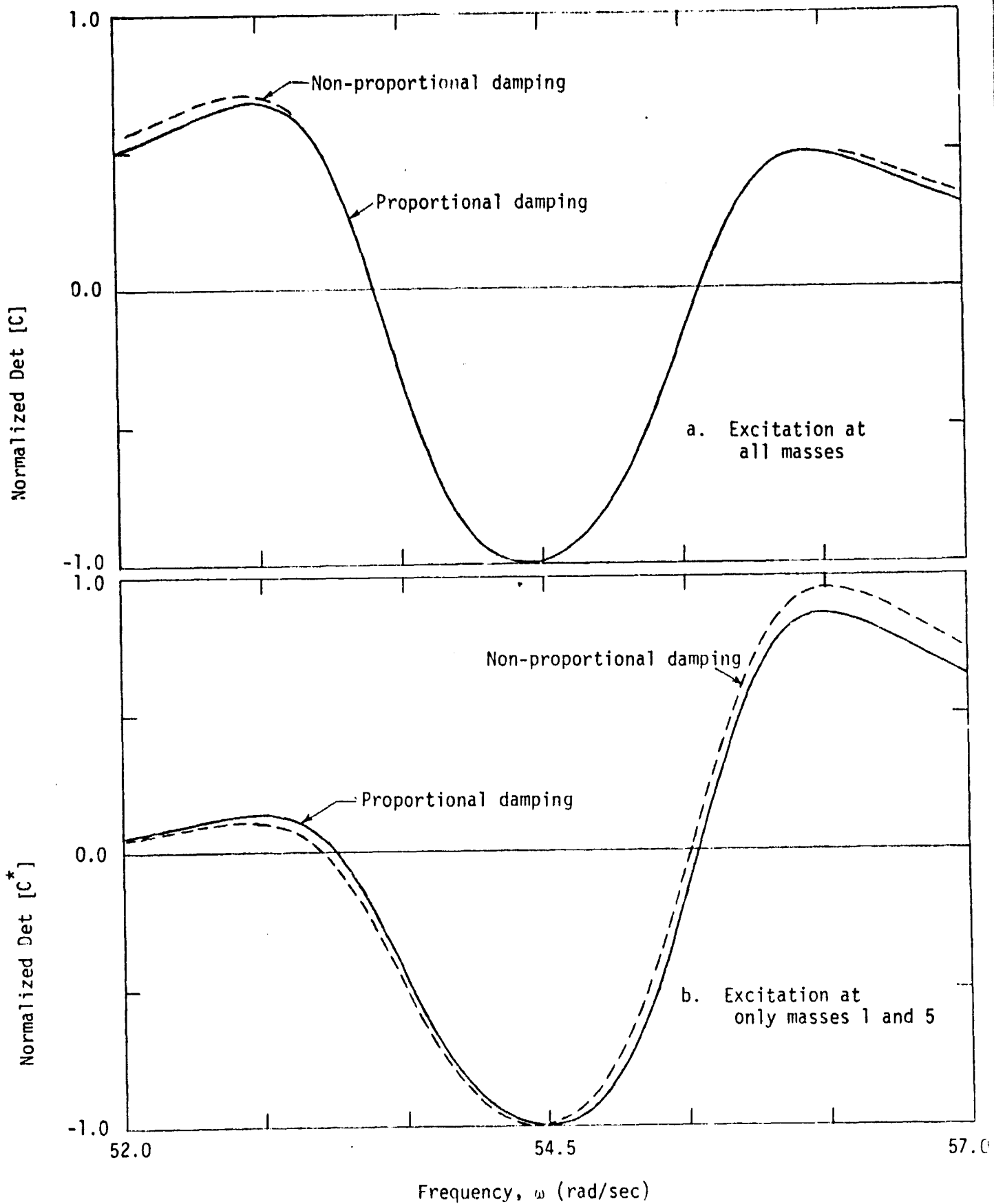


Figure 2 Coincident-Response Determinants



for the two damping cases are barely distinguishable in the frequency range shown, and both have zero crossings exactly at the undamped resonances as is required by theory. Det  $[C^*]$  curves for the two cases are similar but not identical anywhere, as should be expected since they are derived from incomplete transfer function matrices.

Eigenvalue solutions corresponding to the zero crossings of the determinant plots are listed in Tables 1a and b. Each shaker force-amplitude eigenvector is normalized to its largest value. With all masses excited, there are substantial differences between the force-amplitude eigenvectors for proportional and non-proportional damping. This means that the shaker force-amplitude required to isolate a mode is strongly dependent on the specific form of damping, as one would expect. Curiously, this dependence appears to be much weaker with only masses 1 and 5 excited.

#### V. On the Number of Predominant Modes and the Number of Required Shakers

One consequence of our numerical studies is considerable qualitative evidence of a relationship between the number of modes predominant in the motion at a given frequency and the number of shakers required to separate modes in the vicinity of that frequency. This is not a new idea, as Traill-Nash (1958), Bishop and Gladwell (1963), and Asher (1967) have previously discussed it. In fact, Asher proposed a quantitative technique for determining the number of predominant modes by means of a determinant evaluation procedure using transfer-function data. We have tested Asher's technique extensively and have found it to be unreliable.\*

---

\* In contrast, Ibrahim and Mikulcik (1976) employed a similar technique with filtered transient response data and found it quite satisfactory.



a. All masses excited

	Proportional damping		Non-proportional damping	
Zero crossings (rad/sec)	53.50	55.41	53.50	55.41
$F_1$	0.324	1.000	0.208	0.360
$F_2$	0.070	-0.883	0.141	-1.000
$F_3$	1.000	-0.896	1.000	-0.503
$F_4$	-0.234	-0.460	-0.315	-0.348
$F_5$	-0.213	0.736	-0.060	0.116

b. Masses 1 and 5 only excited

	Proportional damping		Non-proportional damping	
Zero crossing (rad/sec)	53.25	55.36	53.16	55.31
$F_1^*$	0.436	0.709	0.444	0.719
$F_5^*$	-1.000	1.000	-1.000	1.000

Tables 1a and b

Coincident-response determinant zero-crossing frequencies and force-amplitude eigenvectors associated with plots in Figure 2.



A concept analogous to the number of predominant modes is that of "a best approximating subspace". This has been analyzed by Cliff (1975) in the context of control theory. Cliff's approach appears to be applicable, with some modifications, to our problem. A detailed description of the theory behind this method is presented in Appendix B. The method involves numerical testing of experimental or simulated experimental transfer function data to determine the number of predominant modes. Within the context of this method, there are several different possible ways to evaluate a given set of data. We are currently conducting numerical studies of these different approaches, and a Master's degree thesis on the subject is in preparation.

If we are able to conclude from this evaluation that Cliff's method provides a reliable test for the number of predominant modes, then we still must determine if there is, in fact, a relationship between this number and the number of shakers required to isolate modes. It is intuitively appealing that there should be such a relationship, but its existence has not been established.

## VI. Other Work in Progress

Our studies thus far of the RFAD method have considered only hysteretically damped structures. However, the results of Ibrahim and Mikulcik (1976) and the continuing work of Ibrahim at NASA LaRC suggest that the damping of real structures might be represented more accurately by a viscous model or possibly a combination hysteretic-viscous model. Therefore, we are extending the capability of our RFAD software to the case of arbitrary viscous damping. The theory employed to solve the forced response problem is that presented in



detail by Hurty and Rubinstein (1964) and used also by Ibrahim and Mikulcik.

We have also begun the design of a simple experimental model patterned after our basic analytical model, Fig. 1. We plan to use aluminum rods as the grid beams and steel rods with axes oriented vertically as the joint masses. To analyze this specimen accurately, we are improving our analytical model to account for the distributed inertia of the aluminum beams and for the rotational inertias of the steel rods. Also, to design closely spaced modes into the specimen, we are revising the mode-merging analysis discussed in Section III and Appendix A to account for the constraints that exist between the translational and rotational inertias of the steel joint masses.

We plan to run modal survey tests on this specimen using excitation equipment on loan to VPI&SU from NASA LaRC and subminiature accelerometers recently acquired by VPI&SU. We also hope to employ a Zonic/Tektronix FFT data processing system, two of which are being purchased by VPI&SU.



## REFERENCES

- Asher, G. W. (1958). A method of normal mode excitation utilizing admittance measurements. Proceedings, National Specialists Meeting, IAS; Dynamics and Aeroelasticity (Institute of the Aeronautical Sciences), pp. 69-76.
- Asher, G. W. (1967). A note on the effective degrees of freedom of a vibrating structure. AIAA Journal, Vol. 5, No. 4, pp. 822-824.
- Bishop, R. E. D. and Gladwell, G. M. L. (1963). An investigation into the theory of resonance testing. Phil. Trans. of the Royal Society of London, Series A, Vol. 255, Mathematical and Physical Sciences, pp. 241-280.
- Bishop, R. E. D. and Johnson, D. C. (1960). The mechanics of vibration. Cambridge University Press.
- Cliff, E. M. (1975). Essential uncontrollability of discrete linear, time-invariant, dynamical systems. Proceedings of the IFAC 75, Part 1, Section 9.3, pp. 1-6. Published by the International Federation of Automatic Control.
- Craig, R. R. and Su, Y. W. T. (1974). On multiple shaker resonance testing. AIAA Journal, Vol. 12, No. 7, pp. 924-931.
- Hurty, W. C. and Rubinstein, M. F. (1964). Dynamics of Structures. Prentice-Hall, Inc.
- Ibrahim, S. R. and Mikulcik, E. C. (1976). The experimental determination of vibration parameters from time responses. The Shock and Vibration Bulletin, Bulletin 46, Part 5, pp. 187-193.
- Mead, D. J. (1970). The existence of normal modes of linear systems with arbitrary damping. Proceedings of a Symposium on Structural Dynamics, Vol. I, Paper No. C5. Loughborough University of Technology, England.
- Stafford, J. F. (1976). A numerical simulation of multiple-shaker modal testing. M. S. thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia.
- Traill-Nash, R. W. (1958). On the excitation of pure natural modes in aircraft resonance testing. J. Aero/Space Sci., Vol. 25, pp. 775-778.
- Zarghamee, M. S. (1968). Optimum frequency of structures. AIAA Journal, Vol. 6, No. 4, pp. 749-750.



## APPENDIX A

### A SIMPLE PROCEDURE FOR PRODUCING CLOSELY SPACED VIBRATION MODES IN AN ANALYTICAL STRUCTURAL MODEL

We consider here the standard n-degree-of-freedom structural dynamic eigenvalue problem,

$$(-\omega_r^2 [m] + [k]) \underline{\phi}_r = \underline{0}, \quad r = 1, 2, \dots, n \quad (1)$$

We assume that the mass and stiffness matrices can be separated into full and diagonal component matrices,

$$\begin{aligned} [m] &= [mf] + [\mu] \\ [k] &= [kf] + [\sigma] \end{aligned} \quad (2)$$

The  $\mu_k$  and  $\sigma_k$  matrix elements correspond to lumped physical members, which often are easily varied in laboratory experimental models as well as analytical models.

It is necessary first to develop expressions for the derivatives of the natural frequencies  $\omega_r$  with respect to the diagonal mass and stiffness elements,  $\mu_k$  and  $\sigma_k$ . The eigensolution for the rth mode of vibration satisfies Eq. 1. Substituting Eq. 2 into Eq. 1 and differentiating it with respect to  $\mu_k$  gives

$$(-2\omega_r \frac{\partial \omega_r}{\partial \mu_k} [m] - \omega_r^2 [1_k]) \underline{\phi}_r + (-\omega_r^2 [m] + [k]) \frac{\partial \underline{\phi}_r}{\partial \mu_k} = \underline{0} \quad (3)$$

where  $[1_k]$  is filled with zeros except for a unit element in the kth diagonal position. We next premultiply Eq. 3 by  $\underline{\phi}_r^T$  and use the definition of the rth generalized mass,  $M_r = \underline{\phi}_r^T [m] \underline{\phi}_r$ . Finally, we simplify the equation by



applying the identity

$$\underline{\phi}_r^T (-\omega_r^2 [m] + [k]) = \underline{0}^T$$

which is simply the transpose of Eq. 1. From this procedure we obtain

$$\frac{\partial \omega_r}{\partial \mu_k} = - \frac{1}{2M_r} \omega_r \phi_{kr}^2 \quad (4)$$

By differentiating Eq. 1 with respect to  $\sigma_k$  and proceeding as above, we may obtain also

$$\frac{\partial \omega_r}{\partial \sigma_k} = \frac{1}{2M_r \omega_r} \phi_{kr}^2 \quad (5)$$

The procedure leading to Eqs. 4 and 5 is due to Zarghamee (1968).

Now, our general objective is to develop an analytical model which is physically reasonable with at least one pair of closely spaced modes. To do this, we may start with a specific numerical structural model and adjust the diagonal terms  $\mu_k$  and  $\sigma_k$ ,  $k = 1, 2, \dots, n$ . To simplify the analysis, let us consider adjustments only to the mass elements,  $\mu_k$ . The requirement of physical reasonableness constrains all  $\mu_k$  to be positive and less than some maximum possible value.

We define vector  $\underline{\mu}$  as the  $n$ -dimensional column of  $\mu_k$  elements. Then the difference  $\Omega$  between any two adjacent natural frequencies may be considered a function of  $\underline{\mu}$ ,

$$\Omega(\underline{\mu}) = \omega_j - \omega_i \quad (6)$$

where we choose  $\omega_j \geq \omega_i$  for this discussion. The Taylor series representation of this functional relation is

$$\Omega(\underline{\mu}) = \Omega(\underline{\mu}_0) + [\nabla \Omega]_{\underline{\mu}_0} \cdot (\underline{\mu} - \underline{\mu}_0) + \dots$$



where  $\underline{\mu}_0$  is an initial state and it is assumed that  $\underline{\mu} - \underline{\mu}_0$  is small so that terms of higher order may be neglected. Hence, with  $\delta\Omega = \Omega(\underline{\mu}) - \Omega(\underline{\mu}_0)$  and  $\delta\underline{\mu} = \underline{\mu} - \underline{\mu}_0$ ,

$$\begin{aligned}\delta\Omega &= [\nabla\Omega]_{\underline{\mu}_0} \cdot \delta\underline{\mu} + \dots \\ &= \underline{g}^T \delta\underline{\mu} + \dots\end{aligned}\quad (7)$$

where

$$g_k = \left[ \frac{\partial\Omega}{\partial\mu_k} \right]_{\underline{\mu}_0} = \left[ \frac{\partial\omega_j}{\partial\mu_k} - \frac{\partial\omega_i}{\partial\mu_k} \right]_{\underline{\mu}_0}$$

Clearly,  $\underline{g}$  may be evaluated with the use of Eq. 4.

To achieve the objective of closely spaced modes, we seek to adjust the masses so that  $\delta\Omega$  is negative. Mass changes  $\delta\underline{\mu}$  must be kept small both to justify the neglect of higher-order terms in Eq. 7 and to maintain physically reasonable mass values. A simple way to maximize  $\delta\Omega$  while holding  $\delta\underline{\mu}$  small is to specify

$$\delta\underline{\mu} = \epsilon \underline{g} \quad (8)$$

where  $\epsilon$  is a dimensional number to be determined. For a given total length of vector  $\delta\underline{\mu}$ , the form specified in Eq. 8 clearly maximizes the inner product in Eq. 7. Substituting Eq. 7 into Eq. 8 gives

$$\delta\Omega = \epsilon \underline{g}^T \underline{g} \quad (9)$$

The change desired in  $\Omega$  may be written as

$$\delta\Omega = f\Omega(\underline{\mu}_0) \quad (10)$$



where  $f$  is a negative decimal number usually ranging from 0 to -0.5. Now  $\epsilon$  may be determined from Eqs. 9 and 10 as

$$\epsilon = f \frac{\Omega(\underline{\mu}_0)}{\underline{g}^T \underline{g}} \quad (11)$$

Hence, the procedure used to bring together two modes of a structural model by adjusting its masses is as follows:

- (i) Calculate the modes of the original configuration, and from those modes calculate with Eq. 4 the frequency-gradient matrix.
- (ii) On the basis of the original modes and the frequency-gradient matrix, select two adjacent modes to be brought together.
- (iii) Using the definition of Eq. 10, select a ratio  $f$ ,  $0 > f > -0.5$ , by which the frequency spacing is to be reduced. If this ratio is too large, the assumption that higher-order terms in Eq. 7 are negligible may be invalid, and this procedure may fail to give satisfactory results.
- (iv) Now calculate the mass change vector which should produce the desired reduction in frequency spacing,

$$\delta \underline{\mu} = \epsilon \underline{g}$$

with  $\epsilon$  given by Eq. 11.

- (v) Calculate  $\underline{\mu}_1 = \underline{\mu}_0 + \delta \underline{\mu}$ , and subsequently calculate the modes for this new configuration. The candidate modes selected in (ii) should now be closer together by about the amount specified in (iii).
- (vi) If it is desirable to bring the candidate modes even closer together, then repeat the procedure as many times as necessary, omitting step (ii) in all iterations after the first.



The theory and procedure presented here can be broadened without much difficulty to account for such situations as specific constraints on or between the mass values  $\mu_k$  and variations of the modes with respect to both lumped masses  $\mu_k$  and lumped stiffnesses  $\sigma_k$ . In the latter case, frequency difference  $\Omega$  would be regarded as a function of both  $\underline{\mu}$  and  $\underline{\sigma}$  and both Eqs. 4 and 5 would be used.



# A METHOD FOR DETERMINING THE NUMBER OF MODES PREDOMINANT IN FORCED SINUSOIDAL RESPONSE

Consider a structural system with  $n$  discrete degrees of freedom. We denote the column matrix of structural response as  $\underline{x}$  and the column matrix of force as  $\underline{f}$ . For steady-state sinusoidal forcing and response at frequency  $\omega$ , we use complex notation,

$$\underline{f} = \text{Re} \left[ \underline{F} e^{i\omega t} \right] = \underline{F} \cos \omega t$$

$$\underline{x} = \text{Re} \left[ \underline{\chi} e^{i\omega t} \right]$$

The complex response vector  $\underline{\chi}$  represents both amplitude and phase of response.

In general, one can employ modal analysis with real or complex modes to express forced response in the series form

$$\underline{\chi} = \sum_{r=1}^n K_r(\omega) \underline{\phi}_r \underline{\phi}_r^t \underline{F}$$

where  $\underline{\phi}_r$  is the mode shape of the  $r$ th mode and  $K_r(\omega)$  is a constant dependent on the generalized mass and damping of the  $r$ th mode. Hence, the complete  $n \times n$  transfer function matrix is

$$[A] = \sum_{r=1}^n K_r(\omega) \underline{\phi}_r \underline{\phi}_r^t$$

And any column of this matrix may be expressed as a linear sum of the  $n$  mode shape vectors,

$$\underline{A}_j = \sum_{r=1}^n K_r(\omega) \phi_{jr} \underline{\phi}_r$$

where  $\phi_{jr}$  is the  $j$ th element of  $\underline{\phi}_r$ .



In order to simplify this discussion, let us restrict attention to situations for which all  $K_r$  and  $\phi_r$  are real. Consider, for example, a system with hysteretic damping which does not couple the undamped normal modes. For such a system, the real part of the admittance matrix, which represents in-phase or coincident response, may be expressed as

$$\text{Re}[A] = [C] = \sum_{r=1}^n K_r(\omega) \phi_r \phi_r^t$$

where

$$K_r(\omega) = \frac{M_r(\omega_r^2 - \omega^2)}{M_r^2(\omega_r^2 - \omega^2)^2 + (M_r \omega_r^2 g_r)^2}$$

in which  $\omega_r$ ,  $M_r$ , and  $g_r$  are the natural frequency, generalized mass, and modal structural damping, respectively, for the  $r$ th mode. With these definitions, each column of the coincident-response matrix can be expressed as a series sum of real  $n$ -dimensional column vectors,

$$C_j = \sum_{r=1}^n K_r(\omega) \phi_{jr} \phi_r$$

In the process of sinusoidal vibration testing, it is usually possible to measure response at  $p$  stations on the specimen, where almost invariably  $p < n$ . Hence, with a single shaker, say the  $j$ th, forcing at frequency  $\omega$ , one measures an incomplete column vector of the admittance matrix,

$$C_j^* = \sum_{r=1}^n K_r(\omega) \phi_{jr}^* \phi_r^*$$

where  $C_j^*$  and  $\phi_r^*$  are  $p$ -dimensional column vectors which lack the  $n-p$  elements of the corresponding  $C_j$  and  $\phi_r$ . Since the  $\phi_r^*$  are  $p$ -dimensional, at most only  $p$  of the  $n$  incomplete mode shape vectors are linearly independent. It seems reasonable



to expect that in general any subset of  $p \phi_r^*$ 's will, in fact, be independent.

At any particular frequency of excitation, we would expect some subset, say  $q$ , of the total modes to be predominant in the response. If this is the case, then each  $C_j^*$  should be a linear sum of the  $q \phi_r^*$ 's corresponding to the predominant modes. This suggests the following procedure for determining the number  $q$ :

- A. If  $k$  shakers and  $p$  motion sensors are available, measure at frequency  $\omega$  the incomplete  $p \times k$  coincident-response matrix whose columns are the  $C_j^*$

$$[C^*] = [C_1^*, C_2^* \dots C_k^*]$$

- B. Now analyze  $[C^*]$  as follows:

- (1) Determine the single  $p$ -dimensional unit vector  $h_1$  which, in a sense to be defined, best represents each column vector  $C_j^*$  as a linear sum, i.e.,

$$C_j^* \approx C_{j1}^1 h_1, \quad j = 1, 2, \dots, k$$

Calculate, in a sense to be defined, the error  $E^1$  resulting from this representation.

- (2) Determine the pair of  $p$ -dimensional orthogonal unit vectors  $h_1$  and  $h_2$  which best represent each  $C_j^*$  as a linear sum, i.e.,

$$C_j^* \approx \sum_{i=1}^2 C_{ji}^2 h_i, \quad j = 1, 2, \dots, k$$

Calculate the error  $E^2$  resulting from this representation.

⋮



(m) Determine the  $m$   $p$ -dimensional orthogonal unit vectors  $\underline{h}_1, \underline{h}_2, \dots, \underline{h}_m$  which best represent each  $\underline{C}_j^*$  as a linear sum.

Calculate the error  $E^m$  resulting from this representation.

⋮

If after  $q$  steps of this procedure we find  $E^q \ll E^{q-1}$  and  $E^q \approx 0$ , then we may reasonably conclude the each  $\underline{C}_j^*$  is, with very small error, a sum of  $q$  orthogonal unit vectors and, therefore, that  $q$  modes predominate in the response at the frequency in question.

If the procedure is carried through  $p$  steps, it is clear that  $E^p = 0$  since the  $p$  independent  $\underline{h}_i$  span  $R^p$ .

The theory required to carry out the calculations in B. above is next developed. To simplify the notation we denote the given incomplete coincident-response matrix as

$$[C^*] = [Z] = [\underline{z}_1, \underline{z}_2, \dots, \underline{z}_k]$$

Let a basis for  $R^p$  consist of  $p$  orthogonal unit vectors  $\underline{h}_1, \underline{h}_2, \dots, \underline{h}_p$ , which are unknown at this point. Hence any real  $p$ -dimensional vector can be expressed as the sum

$$\underline{z}_j = \left( \sum_{i=1}^m + \sum_{i=m+1}^p \right) (\underline{h}_i \cdot \underline{z}_j) \underline{h}_i, \quad j = 1, 2, \dots, k$$

Since the unit  $\underline{h}_i$  are orthogonal,

$$\underline{z}_j \cdot \underline{z}_j = \left( \sum_{i=1}^m + \sum_{i=m+1}^p \right) (\underline{h}_i \cdot \underline{z}_j)^2$$

Hence, we may define the scalar error resulting from representing  $\underline{z}_j$  by only the first  $m$  of the  $\underline{h}_i$  as



$$(e_j^m)^2 = \sum_{i=m+1}^p (h_i \cdot z_j)^2 = z_j \cdot z_j - \sum_{i=1}^m (h_i \cdot z_j)^2$$

It follows that an appropriate total error, defined in the least-squares sense, for all  $z_j$ ,  $j = 1, 2, \dots, k$ , is

$$\begin{aligned} E^m &= \left[ \sum_{j=1}^k (e_j^m)^2 \right]^{1/2} \\ &= \left[ \sum_{j=1}^k z_j \cdot z_j - \sum_{j=1}^k \sum_{i=1}^m (h_i \cdot z_j)^2 \right]^{1/2} \end{aligned}$$

Clearly, the maximum value of the double-summation corresponds to the minimum  $E^m$ , which is the value that we seek. It can be shown that

$$\sum_{j=1}^k (h_i \cdot z_j)^2 = h_i^t [T] h_i$$

where

$$[T] = [Z][Z]^t$$

$[T]$  is a  $p \times p$  real, symmetric, positive-semidefinite matrix, so it has  $p$  non-negative real eigenvalues  $\lambda_i$  and  $p$  corresponding real orthogonal eigenvectors  $\psi_i$ . It is convenient to take the  $\psi_i$  to be unit vectors. It can be shown that the maximum values of the quadratic form involving  $[T]$  are

$$\{h_i^t [T] h_i\}_{\max} = \psi_i^t [T] \psi_i = \lambda_i$$

Hence, in order to minimize  $E^m$  we take  $h_i = \psi_i$ ,  $i = 1, 2, \dots, p$ , and we obtain

$$\{E^m\}_{\min} = \left[ \sum_{j=1}^k z_j \cdot z_j - \sum_{i=1}^m \lambda_i \right]^{1/2}$$



We can now reiterate and summarize the method for determining the number  $q$  of predominant modes at a given frequency:

- (i) Using experimental or analytical data, assemble the  $p \times k$  incomplete coincident-response matrix  $[C^*]$  based upon  $k$  shaker locations and  $p$  sensor locations.
- (ii) Form the  $p \times p$  matrix

$$[T] = [C^*][C^*]^t$$

- (iii) Calculate the eigenvalues of  $[T]$ ,  $\lambda_i$ ,  $i = 1, 2, \dots, p$ .
- (iv) For  $m = 1, 2, \dots, p$ , calculate the error

$$E^m = \left[ \sum_{j=1}^k C_j^* \cdot C_j^* - \sum_{i=1}^m \lambda_i \right]^{1/2}$$

If there are in fact  $q < p$  predominant modes, then this should be indicated by the observations that  $E^q < < E^{q-1}$  and  $E^q \approx 0$ .